Introduction

In 2014, Solvency II, a new regulatory framework, will come into effect for European insurance companies. The objective is to protect insurance policyholders and beneficiaries. This whitepaper focuses on one of the most important components of the Solvency II Capital Requirements (SCR) market risk module, the interest rate risk. We present and compare six different interest rate models to find and set up a simple, accurate and robust interest rate model satisfying the Solvency II requirements for measuring interest rate risk. The interest rate models used are the equilibrium models, the Vasicek, CIR and the Two-Factor Vasicek model, and the arbitrage free models, the Hull-White and Two-Factor Arbitrage Free Vasicek model and the Nelson-Siegel model. The key requirement under Solvency II concerning the SCR for market risk is that insurance companies need to maintain sufficient capital to be able to meet their obligations over the next 12 months with a probability of at least 99.5%.

We used different performance measures to investigate which model performs optimal under Solvency II and thus satisfies the requirements for an interest rate model under the Solvency II. We depend the choice of the optimal interest rate model on which performance measure is preferred.

Data

Daily DNB (Dutch Central Bank) euro swap zero rate data with maturities 1 till 60 year is used. This covered only the period 2001-2010. For this reason, also data from an alternative source covering a longer period is taken into account, i.e. from the Bundesbank. These are monthly zero rates covering the period 1986-2012 for the maturities 1 till 15 year.

The following “stylized facts of the yield curve” as described by Diebold and Li (2006) are confirmed for the interest rate data:

- The average yield curve over time is increasing and concave
- The yield curve can take on a variety of shapes (upward sloping, downward sloping, humped, inverted humped, S-shapes)
- Yield dynamics are (very) persistent (high auto-correlations)
- Yields for long maturities are more persistent than yields for shorter maturities
- Short end of the yield curve is more volatile than the long end of the curve
- Yields for different maturities have high cross-correlations
Interest Rate Models

An interest rate model is a model that describes the evolution of a zero curve through time. There are many competing interest rate models available. A difference can be made between equilibrium models and no-arbitrage models. Furthermore, each model has its own pro’s and con’s. We will describe both equilibrium and no-arbitrage models.

Equilibrium models, as described by Hull (2003) start with assumptions about economic variables and derive a process for the short rate, which means that the current term structure of interest rates hence is an output rather than an input in the model. Therefore, they are also called endogenous term-structure models. The (instantaneous) short rate at time t is the rate that applies to an infinitesimally short period of time at time t.

We will explain some popular equilibrium models. Namely, the Vasicek and the Cox, Ingersoll and Ross (CIR) model. These models are one-factor models, which have several shortcomings, such as that they assume that the interest rates are perfectly correlated between different maturities. To overcome this problem, we will also describe the Two-Factor Vasicek model.

A no-arbitrage model is a model designed to be exactly consistent with today’s term structure of interest rates. The difference between equilibrium and a no-arbitrage model is that in an equilibrium model, today’s term structure of interest rates is an output. In a no-arbitrage model, today’s term structure of interest rates is an input. This means that while constructing the model we take the observed actual rates and estimate the unobserved rates. We will elaborate on the Hull and White and Two-Factor Arbitrage Free Vasicek model.
The Vasicek model

The Vasicek model is a method which is widely used, easy to understand and incorporates mean reversion. A drawback of the Vasicek model is that the volatility of the short rate is constant. Another disadvantage of the model is that it permits negative interest rates in theory.

The Vasicek model assumes that for the ‘short rate’ $r_t$ follows an Ornstein-Uhlenbeck process:

$$dr_t = \alpha (\beta - r_t) dt + \sigma dW_t, \quad r(0) = r_0$$

Where

- $dr_t$ is the change in interest rate
- $\alpha$ measures the speed of mean reversion
- $\beta$ is the ‘steady state mean’, to which the process tends to revert in the long run.
- $t$ Time
- $\sigma$ A measure of the process volatility
- $dW_t$ The standard Brownian motion, so $W_t \sim N(0, dt)$

The CIR model

The Cox, Ingersoll and Ross (CIR) equilibrium model is an alternative to the Vasicek model. The CIR model always produces nonnegative interest rates. Both the Vasicek and CIR model incorporate the mean-reversion feature.

The CIR model assumes that the ‘short rate’ $r_t$ follows a process of the type:

$$dr_t = \alpha (\beta - r_t) dt + \sigma \sqrt{r_t} dW_t, \quad r(0) = r_0$$

As we can see, the model has the same mean reverting drift as the Vasicek model. But the volatility part of the model is different. In the CIR model the volatility of the short rate is assumed to be proportional to the level of interest rates through $\sqrt{r_t}$. The model hence allows more variability at times of high interest rates and less variability when rates are low. The CIR model thus keeps rates of interest positive.
The Two-Factor Vasicek model

We decided the number of factors with the use of Principal Components Analysis. This analysis showed that two-factor models can explain up to 95% variation in the yield curve.

The ‘short rate’ $r_t$ of a Two-Factor Vasicek model follows a process of the type

\[
\begin{align*}
r_t &= x_t + y_t, r(0) = x_0 + y_0 \\
dx_t &= \alpha_x (\beta_x - x_t) dt + \sigma_x dW^1_t, x(0) = x_0 \\
dy_t &= \alpha_y (\beta_y - y_t) dt + \sigma_y dW^2_t, y(0) = y_0
\end{align*}
\]

Where $(W_1, W_2)$ is a two-dimensional Brownian motion with instantaneous correlation $\rho$ as from

\[
dW^1_t dW^2_t = \rho dt
\]

where $-1 \leq \rho \leq 1$.

The $x_t$ and $y_t$ processes are modeled as the short-rate and the long-rate in order to explain the variation in the zero rates better. Since both processes are modeled identically, the question of which process is modeled as the short rate or the long rate is not important.

The Nelson-Siegel model

Diebold and Li (2006) gave the following representation to the three-factor term structure model proposed by Nelson and Siegel (1987):

\[
y(t, \tau) = \beta_{1t} + \beta_{2t} \left( \frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right) + \beta_{3t} \left( \frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_3 \tau} \right).
\]

Where $\beta_{1t}, \beta_{2t}, \beta_{3t}$ and $\lambda_1$ are time-varying parameters. $y(t, \tau)$ denotes the yield at observed time $t$ with maturity time $\tau$. 
The Hull and White model

The Hull and White model is an extension of the Vasicek model that provide an exact fit to the initial term structure. The following ‘short rate’ process is considered

\[ dr(t) = (\theta(t) - \alpha r(t))dt + \sigma dW(t) \]

Where the time-dependent mean parameter \( \theta \) is chosen in order to exactly fit the term structure of interest rates. This model is an arbitrage-free model, as the model ensures an exact fit to the current term structure.

The \( \theta(t) \) function can be calculated form the initial term structure:

\[ \theta(t) = \frac{\partial f^M(0,t)}{\partial \tau} + \alpha f^M(0,t) + \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \]

Where \( f^M(0,t) \) denotes the market instantaneous forward rate at time 0 for the maturity \( \tau \), i.e.,

\[ f^M(0,\tau) = \frac{\partial \ln P^M(0,\tau)}{\partial \tau} \]

The Two-Factor (AF) model

We did also analyze the arbitrage-free version of the Two-Factor Vasicek model. The Two-Factor Arbitrage-Free Vasicek model the ‘short rate’ \( r(t) \) follows a process of the type

\[ r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0 \]

Where the processes \( \{x(t) : t \geq 0\} \) and \( \{y(t) : t \geq 0\} \) satisfy

\[ dx(t) = -\alpha x(t)dt + \sigma_x dW_1(t), \quad x(0) = 0 \]
\[ dy(t) = -\alpha y(t)dt + \sigma_y dW_2(t), \quad y(0) = 0 \]

Where \((W_1, W_2)\) is a two-dimensional Brownian motion with instantaneous correlation \( \rho \) such that

\[ dW_1(t)dW_2(t) = \rho dt \]

Where \( r_0, \alpha, \sigma_x, \sigma_y, \rho \) are positive constants and \(-1 \leq \rho \leq 1\). The function \( \varphi \) is deterministic and \( \varphi(0)=r_0 \).
Results
We considered the following different performance measures in order to investigate which model performs optimal under Solvency II:

- **The fit of the yield curve**
  Comparing the yield curve obtained by the different models to the yield curves determined by the data.

- **Benchmarking**
  Comparing the interest rate stress factors (upward and downward) obtained by the different models in this research with those proposed by CEIOPS and QIS4.

- **Risk Simulation**
  Quantifying the risk which investors face when they sell their bonds before using the Value-at-Risk (VaR) method.

- **Robustness**
  The effect of ±10% and ±20% changes in the parameters of the different models.

- **Adequateness**
  Evaluating the models by applying them to historical data (‘backtesting’). By applying the model at time $t$ to forecast the 1 in 200 year upward and downward stress events for time $t+1$. This is applied this to a certain portfolio. These values are compared with the observed interest rates at time $t+1$ to verify if the true value was in the confidence interval.

- **Simplicity**
  A complex model is harder to understand, usually less stable and may involve more parameters requiring a large set of data. By simple we mean the number of factors and parameters in the model.

This research does not lead to one optimal interest rate model. The choice of the optimal interest rate model depends on which performance measure is preferred. If the fit of the yield curve is an important aspect then we recommend the arbitrage free models or the Nelson-Siegel model. In case the company cares more about the similarity between the stress factors of the model and those proposed by CEIOPS and QIS4, then the Hull-White or Vasicek model is recommended. If risk plays an important factor, then the CIR model is the most profitable. When the company wants an robust model then the Two-Factors Vasicek, Hull-White or Two-Factor AF Vasicek model is recommended. If the adequateness of the model under Solvency II is an weighty aspect, then we advise the Nelson-Siegel, Hull-White or Two Factor AF Vasicek model. If one wants a simple model then we recommend the Vasicek, CIR or Hull-White model. The model that satisfies the most performance measures is the Hull and White model.
RiskQuest is an Amsterdam based consultancy firm specialised in risk models for the financial sector. The importance of these models in measuring risk has strongly increased, supported by external regulations such as Basel II/III and Solvency II.

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