The optimal Nelson-Siegel model within the Solvency II framework
Introduction

Interest rates are a key value driver to financial institutions. For this reason they are very interested in the ability to correctly understand, model and predict the dynamics of interest rates at different maturities, i.e., yield curves. Knowledge of the yield curves and their dynamics is crucial in many topics in finance and economics like: asset pricing, derivative pricing, risk management, portfolio allocation, choosing the financing structure of a company and valuation of economic capital. To ensure the continuity of their businesses, financial institutions have to hold a certain amount of money - a capital requirement - to overcome unexpected adverse events. In 2014 a new regulation framework called Solvency II, was planned to come into act for all insurance companies within Europe, however the exact time of introduction is delayed until further notice. It will provide a general framework to harmonize standards for assets and liabilities valuation across the European Union and intends to increase stability within the financial sector. This new legislation dictates a Solvency Capital Requirement (SCR). Solvency II also provides a standard framework for a wide variety of risk management purposes and reporting standards to European insurance companies.

This white paper identifies an optimal and appropriate Nelson-Siegel model specification to determine the capital requirement for interest rate risk within the Solvency II framework. We mainly focus on the 12-month ahead (density) forecasts of the yield curves. Various model specifications are tested using different estimation and forecasting techniques. To get a complete assessment of the available econometric theories, Bayesian inference theory is used. Solvency II focuses on forecasts consisting of a full distribution, which is where the Bayesian inference theory can prove its power.

Four model specification

We make use of 4 different well-documented Nelson-Siegel model specifications.

Standard dynamic Nelson-Siegel (DNS-3)

Diebold and Li (2006) proposed the following model to describe the yield curve at time \( t \) at any maturity \( \tau \). (It is an adjustment to the original Nelson-Siegel model)

\[
y_t(\tau) = L_t + S_t \left[ \frac{1 - \exp \left( -\frac{\tau}{\lambda} \right)}{\left( \frac{\tau}{\lambda} \right)} \right] + C_t \left[ \frac{1 - \exp \left( -\frac{\tau}{\lambda} \right)}{\left( \frac{\tau}{\lambda} \right)^2} - \exp \left( -\frac{\tau}{\lambda} \right) \right].
\]

These three components provide enough flexibility in order to capture the most commonly seen shapes in yield curves like a monotonic, humped or S-type shape. 95% of the yield variation is captured by these 3 factors. This dynamic version with 3-factors is henceforth called Dynamic Nelson-Siegel (DNS-3). An important and convenient insight is the economic interpretation of the three factors as level, slope and curvature factors respectively.
Dynamic Nelson-Siegel 2 factor (DNS-2)
Since 95% of the variation in yield is explained by the first three principal components, Nelson and Siegel proposed three components. However, not much contribution is made by the third component as shown by Litterman and Scheinkman (1991). Diebold, Piazzesi and Rudebusch (2005) propose the following model, henceforth called DNS-2:

\[ y_t(\tau) = L_t + S_t \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_1}\right)}{\left(\frac{\tau}{\lambda_1}\right)^3} \right]. \]

The DNS-2 model should be able to explain many of the variation in the yields and only consists of 2 factors, which could be beneficial for estimation purposes.

Adjusted dynamic Nelson-Siegel Svensson (DNS-S)
A widely used model for the term structure is the model proposed by Svensson (1994). The model is proposed to add more flexibility by including an extra curvature component with a different decay parameter. The Dynamic Nelson-Siegel Svensson (DNS-S) model to fit the yield curve is given by:

\[ y_t(\tau) = L_t + S_t \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_1}\right)}{\left(\frac{\tau}{\lambda_1}\right)^3} \right] + C_{1,t} \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_2}\right)}{\left(\frac{\tau}{\lambda_2}\right)^3} \right] - \exp \left( -\frac{\tau}{\lambda_4} \right) + C_{2,t} \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_3}\right)}{\left(\frac{\tau}{\lambda_3}\right)^3} \right]. \]

The main advantage is that the fourth component is able to fit more easily yield curves with more than one local maximum or minimum. This factor is a second ‘medium term’ component so these local maxima or minima are best fitted in the range of medium term maturities.

Bjork & Christensen four-factor model (DNS-4)
The adjustment as proposed by Svensson focused mainly on additional flexibility to the model and improving fitting performances at medium maturities. Bjork & Christensen proposed a different adjustment to the classic DNS model. Their adjustment is to add an extra component for the short-term maturities. The Bjork & Christensen four-factor model (DNS-4) is given by:

\[ y_t(\tau) = L_t + S_t \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_4}\right)}{\left(\frac{\tau}{\lambda_4}\right)^3} \right] + C_{1,t} \left[ \frac{1-\exp\left(-\frac{\tau}{\lambda_4}\right)}{\left(\frac{\tau}{\lambda_4}\right)^3} - \exp \left( -\frac{\tau}{\lambda_4} \right) \right] + C_{2,t} \left[ \frac{1-\exp\left(-\frac{2\tau}{\lambda_4}\right)}{\left(\frac{2\tau}{\lambda_4}\right)^3} \right]. \]
Models to capture the dynamics of the yields
The models as described above can be estimated using different techniques. We incorporated OLS for its simplicity and NLS for its additional flexibility. Autoregressive models are used to capture the dynamics of the yields. We use the AR(1) model as the benchmark model. To model the possible cross-correlation in the factors we can add a lot of flexibility using the VAR(1) model. All possible combinations using OLS/NLS/AR(1)/VAR(1) are taken into account.

Data description
The data used in this paper all consist of end-of-month observations of Government bonds yield curves, which is consistent with the Solvency II regulations. The regulator prescribes the use of appropriate interest rate data, which meet certain requirements, i.e., qualities. These requirements are summarized by: Risk free, reliable, liquid, unbiased and independent. This paper makes use of three different datasets of government bonds to be able to identify a general conclusion. See Table 1.

<table>
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<th>Type</th>
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<td>1970:1 – 2000:12</td>
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<td>1985:1 – 2013:1</td>
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<td>1 year – 20 years</td>
<td>Deutsche Bundesbank</td>
<td>1986:6 – 2013:2</td>
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</table>

Table 1

Forecasting performance within the Solvency II framework
To test significant outperformance of model i against benchmark model j within a specific time interval t = 1,…,N the Diebold-Mariano (1995) test is implemented. The test is based on an average loss differential and divided by the square root of its asymptotical variance.

What becomes clear from the vast maturity of forecasts in every dataset is that the more parsimonious DNS-3 model outperforms all other models. The advantage of the NLS method to add more flexibility to the model is only significantly beneficial in all datasets for the more parsimonious DNS-3 AR(1) model. A reasonable and safe conclusion is that the NLS method should not be combined with VAR(1) dynamics and only for the more parsimonious models like the 2 and the 3 factor models. The combination of NLS and VAR(1) turns out to be too numerically unstable, especially for risk management purposes.

There is no VAR(1) specification at all that improves the forecasting results compared to the AR(1) specification. To state it even stronger, the VAR(1) specification significantly worsens the forecasts. In the three datasets there is no model
specification using VAR(1) dynamics that is beneficial to the forecasts. De Pooter (2007) showed that even with the presence of highly cross-correlated factors the VAR(1) model often shows very poor forecasting performance, presumably due to overfitting issues.

**Density analysis using Bayesian inference**

To capture the uncertainty around the cross section fit and the point forecasts of yields we are strongly benefitted with knowledge of the yield distribution. Instead of only fixed point forecasts of model parameters as produced using classical econometric inference, Bayesian inference provide some advantages over classical inference:

- It provides better interpretable and more intuitive answers.
- Bayesian methods make use of all available information.
- Bayesian methods have no reliance on any asymptotic approximation theory
- Bayesian techniques are particularly well suited for decision-making

As described in Van Oest, Hoogerheide & Van Dijk (2003), in the Bayesian treatment of inference, the parameters are not treated as fixed unknown constants which have to be recovered, but the model parameters are considered to be random variables themselves. In Bayesian analysis the only way of thinking of a model parameter is as a random variable, which can take on a wide range of values. The initial beliefs of parameters may be summarized in a density. This density is called a prior density since it reflects beliefs about the model parameters prior to actually observing the data. Beliefs about the parameters are updated with every new available data observation. The updated beliefs are depicted in the posterior density, the density of the parameters post observing the data, see Table 2. Modern Bayesian analysis is heavily relying on various simulation techniques with numerous evaluations of integrals and complex density functions. The most popular and well-known simulation techniques are importance sampling and Markov Chain Monte Carlo methods such as the Gibbs sampler and the Metropolis-Hastings algorithm.

![Image](image_url)

| p(θ|y)  | α | p(θ) | x | p(y|θ) |
|--------|----|------|---|--------|
| Posterior density | α | prior density | x | likelihood |
| Beliefs after observing the data | Beliefs before observing the data | Influence of the data |

*Table 2 - Learning principle of Bayesian analysis*
Using these sampling methods we are able to simulate the posterior density at different months. The 4 factor models seem to be too complex to work with in this Bayesian environment. Therefore we can only test the 2 and 3 factor models. The solid/dotted density represents the DNS-2/DNS-3 model respectively, where the vertical black line represents the actual interest rate at that maturity. Figure 1 shows the simulated density forecasts of 5 maturities of the yield at December 2012. These forecasts are 12-months ahead, so they are made at December 2011.

Figure 1 a,b,c,d,e,f - Forecasted yield densities for December 2012 in the FED dataset. The black vertical lines represents the actual realized interest rate at the selected maturities. The solid densities represents the DNS-2 model with its best approximated normal counterpart. The dotted densities represents the DNS-3 model with its best approximated normal counterpart. Figure 1f, shows the entire yield curve at December 2012.

From Figure 1 it is quite clear that the DNS-3 model is again outperforming the DNS-2 model. The DNS-3 model performs better in terms of simulated modes closer to the actual yield and lower simulated variance. This is observed in every dataset for every selected forecasted period. Figure 1 shows that the forecasts are highly improved due to the additional flexibility imposed by the additional curvature component of the DNS-3 model. The inclusion of this additional component also reduces the uncertainty (variance) in the forecasted densities, which is obviously an attractive feature.
Concluding remarks

The classical inference theory shows clearly that the more parsimonious models outperform the more complex models. The additional flexibility of using the NLS method is not beneficial compared to the OLS method. The VAR(1) model to capture possible cross-correlation heavily deteriorates the performances. These conclusions are confirmed and shown in the density analysis using Bayesian inference. The modern sampling algorithms are not even capable to handle the more complex models consisting of 4 factors. The DNS-3 AR model is clearly outperforming the other models in terms of accurate mode/mean forecasting and in terms of lowest forecasting variance at forecasts of 12-months ahead.
RiskQuest is an Amsterdam based consultancy firm specialised in risk models for the financial sector. The importance of these models in measuring risk has strongly increased, supported by external regulations such as Basel II/III and Solvency II.

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The business areas that we cover are lending, financial markets and insurance. In relation to the models, we provide advice on: Strategic issues; Model development; Model validation; Model use.